Online appendix for the paper

Relative Expressiveness of Defeasible Logics II

published in Theory and Practice of Logic Programming

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submitted 10 April 2013; revised 23 May 2013; accepted 23 June 2013

Appendix

This appendix contains the inference rules for the logics in DL, proofs of results in the body of the paper (Maher 2013), and some examples. Theorems, Lemmas, or Examples numbered 1–14 refer to items in the body of the paper. Larger numbers refer to items in this appendix.

Inference Rules for DL

For every inference rule $+d$ there is a closely related inference rule $-d$ allowing to infer that some literals $q$ cannot be consequences of $D$ via $+d$. The relationship between $+d$ and $-d$ is described as the Principle of Strong Negation (Antoniou et al. 2000). These inference rules are placed adjacenty to emphasize this relationship.

Some notation in the inference rules requires explanation. Given a literal $q$, its complement $\sim q$ is defined as follows: if $q$ is a proposition then $\sim q = \neg q$; if $q$ has form $\neg p$ then $\sim q$ is $p$. We say $q$ and $\sim q$ (and the rules with these literals in the head) oppose each other. $R_s$ ($R_ad$) denotes the set of strict rules (strict or defeasible rules) in $R$. $R[q]$ ($R_s[q]$, etc) denotes the set of rules (respectively, strict rules) of $R$ with head $q$. Given a rule $r$, $A(r)$ denotes the set of literals in the body (or antecedent) of $r$.

$$+\Delta) +\Delta q \in T_D(E) \text{ iff either }$$

1. $q \in F$; or
2. $\exists r \in R_s[q]$ such that $\forall a \in A(r), +\Delta a \in E$

$$-\Delta) -\Delta q \in T_D(E) \text{ iff }$$

1. $q \notin F$; and
2. $\forall r \in R_s[q]$ $\exists a \in A(r), -\Delta a \in E$

$$+\partial) +\partial q \in T_{D}(E) \text{ iff either }$$

1. $+\Delta q \in E$; or
2. The following three conditions all hold.
   1. $\exists r \in R_{ad}[q]$ $\forall a \in A(r), +\partial a \in E$, and
   2. $-\Delta \sim q \in E$, and
   3. $\forall s \in R[\sim q]$ either
      1. $\exists a \in A(s), -\partial a \in E$; or
      2. $\exists t \in R_{ad}[q]$ such that
         1. $\forall a \in A(t), +\partial a \in E$, and
         2. $t > s$.

$$-\partial) -\partial q \in T_{D}(E) \text{ iff }$$

1. $-\Delta q \in E$; and
2. either
   1. $\forall r \in R_{ad}[q]$ $\exists a \in A(r), -\partial a \in E$; or
   2. $+\Delta q \in E$; or
   3. $\exists s \in R[\sim q]$ such that
      1. $\forall a \in A(s), +\partial a \in E$, and
      2. $\forall t \in R_{ad}[q]$ either
         1. $\exists a \in A(t), -\partial a \in E$; or
         2. not($t > s$).
We now turn to proofs of results in the body of the paper, and some examples. This part of the
sequence of
All simulation proofs (of
δ∗ ∈ T
q
) +
.1) θΔq ∈ E, or
.2) ∃r ∈ Rsd[q] such that
.1) ∀a ∈ A(r), +δ∗a ∈ E, and
.2) −Δ q ∈ E, and
.3) ∃s ∈ R[~q] either
.1) θa ∈ A(s), −δ∗a ∈ E; or
.2) r > s.

+δ) +δq ∈ T
q
E iff either
.1) +Δq ∈ E; or
.2) The following three conditions all hold.
.1) ∃r ∈ Rsd[q] ∀a ∈ A(r), +δa ∈ E, and
.2) −Δ q ∈ E, and
.3) ∃s ∈ R[~q] either
.1) θa ∈ A(s), −σa ∈ E; or
.2) ∃a ∈ Rsd[q] such that
.1) ∀a ∈ A(t), +δa ∈ E, and
.2) t > s.

+σ) +σq ∈ T
q
E iff either
.1) +Δq ∈ E; or
.2) ∃r ∈ Rsd[q] such that
.1) ∀a ∈ A(r), +σa ∈ E, and
.2) ∃s ∈ R[~q] such that
.1) θa ∈ A(s), −δa ∈ E; or
.2) not(s > r).

+δ∗) +δ∗q ∈ T
q
E iff either
.1) +Δq ∈ E; or
.2) ∃r ∈ Rsd[q] such that
.1) ∀a ∈ A(r), +δ∗a ∈ E, and
.2) −Δ q ∈ E, and
.3) ∃s ∈ R[~q] either
.1) θa ∈ A(s), −δ∗a ∈ E; or
.2) r > s.

Proofs of results

We now turn to proofs of results in the body of the paper, and some examples. This part of the
appendix has the same structure as the paper itself, to make access easier.

All simulation proofs (of DL(d1) by DL(d2), say) have two parts: first we show every conse-
quence of D+A in DL(d1) has a corresponding consequence of T(D)+A in DL(d2), and then
we show that every consequence of \( T(D)+A \) in \( \text{DL}(d_2) \) in the language of \( D+A \) has a corresponding consequence of \( D+A \) in \( \text{DL}(d_1) \). In both cases the proof is by induction on the level \( n \) of \( T \uparrow n \) where \( T \) combines the functions in the inference rules for \( \pm d_1 \) and \( \pm \Delta \) for \( D+A \) in the first part, and combines the functions in the inference rules for \( \pm d_2 \) and \( \pm \Delta \) for \( T(D)+A \) in the second part. The induction hypothesis for the first part is: for \( k \leq n \), if \( \alpha \in T_{D+A} \uparrow n \) then \( T(D)+A \vdash \alpha' \), where \( \alpha' \) is the counterpart, in \( \text{DL}(d_2) \), of \( \alpha \). For the second part it is: for \( k \leq n \), if \( \alpha \in \Sigma \) and \( \alpha \in T_{T(D)+A} \uparrow n \) then \( D+A \vdash \alpha' \), where \( \alpha' \) is the counterpart, in \( \text{DL}(d_1) \), of \( \alpha \). Since \( T_{D} \uparrow 0 = 0 \) the induction hypothesis is always valid for \( n = 0 \).

Throughout this appendix, if \( r \) is a rule then \( B_r \) refers to the body of that rule. For brevity, we write \(+dB\), where \( B \) is a set of literals, to mean \( \{ +dq \mid q \in B \} \).

### Blocked Ambiguity Simulates Propagated Ambiguity

The facts and strict rules of \( D+A \) and \( T(D)+A \) are the same, except for rules for \( \text{strict}(q) \) in \( T(D)+A \). However \( \text{strict}(q) \) is not used in any other strict rule. Consequently, for any addition \( A \), \( D+A \) and \( T(D)+A \) draw the same strict conclusions in \( \Sigma(D+A) \). Furthermore, these conclusions are reflected in the defeasible conclusions of \( \text{strict}(q) \).

#### Lemma 15

Let \( A \) be any defeasible theory, and let \( \Sigma \) be the language of \( D+A \). Then, for every \( q \in \Sigma \),

- \( D+A \vdash +\Delta q \) iff \( T(D)+A \vdash +\Delta q \)
  - iff \( T(D)+A \vdash +\partial^* \text{strict}(q) \) iff \( T(D)+A \vdash -\partial^* \neg \text{strict}(q) \)
- \( D+A \vdash -\Delta q \) iff \( T(D)+A \vdash -\Delta q \)
  - iff \( T(D)+A \vdash -\partial^* \text{strict}(q) \) iff \( T(D)+A \vdash +\partial^* \neg \text{strict}(q) \)

#### Proof

The proof of \( D+A \vdash \pm \Delta q \) iff \( T(D)+A \vdash \pm \Delta q \) is straightforward, by induction on length of proofs.

In the inference rule for \( +\partial^* \text{strict}(q) \), clause .2.3 must be false, by the structure of the rules in part 3 of the transformation. Consequently, we infer \( +\partial^* \text{strict}(q) \) iff we infer \( +\Delta \text{strict}(q) \), which happens iff we infer \( +\Delta q \) since there is only the one rule for \( \text{strict}(q) \). Similarly, clause .2.3 of the inference rule for \( -\partial^* \text{strict}(q) \) is true, so we infer \( -\partial^* \text{strict}(q) \) iff we infer \( -\Delta \text{strict}(q) \), which happens iff we infer \( -\Delta q \) since there is only the one rule for \( \text{strict}(q) \).

In the inference rule for \( -\partial^* \neg \text{strict}(q) \), clause .2.1 is false because the body of \( nstr(q) \) is empty, and clause .2.3 is false because \( nstr(q) >' str(q) \). Thus we infer \( -\partial^* \neg \text{strict}(q) \) iff we infer \( +\Delta \text{strict}(q) \). Finally, in the inference rule for \( +\partial^* \neg \text{strict}(q) \), clause .1 is false, because there is no fact or strict rule for \( \neg \text{strict}(q) \), and clauses .2.1 and .2.3 are true (the latter because \( nstr(q) >' str(q) \)). Thus, we can infer \( +\partial^* \neg \text{strict}(q) \) iff we can infer \( -\Delta \text{strict}(q) \).

This lemma establishes that strict provability \( (\pm \Delta) \) from \( D+A \) in \( \text{DL}(\delta^*) \) is captured in \( \text{DL}(\partial^*) \) by the transformation defined above, no matter what the addition \( A \). We now show that \( \text{DL}(\partial^*) \) can simulate the behaviour of both \( \delta^* \) and \( \sigma^* \) with respect to addition of facts.

#### Lemma 16

Let \( D \) be a defeasible theory, \( T(D) \) be the transformed defeasible theory as described in Definition 3, and let \( A \) be a modular set of facts. Let \( \Sigma \) be the language of \( D+A \) and let \( q \in \Sigma \). Then
Proof

Suppose $+\sigma^* q \in T_{D+A} \uparrow (n+1)$. Then, by the $+\sigma^*$ inference rule, there is a strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$, such that $+\sigma^* B_r \subseteq T_{D+A} \uparrow n$, and for every rule $s$ in $D$ for $\sim q$ either there is a literal $b$ in the body of $s$ such that $-\delta^* b \in T_{D+A} \uparrow n$ or $s \not> r$. Hence, by the induction hypothesis, there is a strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$ such that $T(D)+A \vdash +\delta^* \text{supp}(b)$ for each $b \in B_r$, and for every rule $s$ in $D$ for $\sim q$ either there is a literal $b$ in the body of $s$ such that $T(D)+A \vdash -\delta^* b$ or $s \not> r$. Then $T(D)+A \vdash +\delta^* \text{supp}_D(r)$ and for for every rule $s$ in $D$ for $\sim q$ with $s > r$ $T(D)+A \vdash -\delta^* B_s$, and hence $T(D)+A \vdash +\delta^* -\sigma(q)$. Combining these two conclusions, and given that there is no rule for $-\text{supp}(q)$, we have $T(D)+A \vdash +\delta^* \text{supp}(q)$.

Suppose $+\delta^* q \in T_{D+A} \uparrow (n+1)$. Then, by the $+\delta^*$ inference rule, there is a strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$, such that $+\delta^* B_r \subseteq T_{D+A} \uparrow n$, and for every rule $s$ in $D$ for $\sim q$ where $r \not> s$, there is a literal $b$ in the body of $s$ such that $-\sigma^* b \in T_{D+A} \uparrow n$. Hence, by the induction hypothesis, there is a strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$ such that $T(D)+A \vdash +\sigma^* b$, $T(D)+A \vdash -\Delta q$, and for every rule $s$ in $D$ for $\sim q$ where $r \not> s$, there is a literal $b$ in the body of $s$ such that $T(D)+A \vdash -\sigma^* \text{supp}(b)$. By Lemma 15, $T(D)+A \vdash -\sigma^* \text{strict}(\sim q)$. By repeated application of the $-\sigma^*$ inference rule we have $T(D)+A \vdash -\sigma^* \text{supp}_D(s)$ for each $s$, and then $T(D)+A \vdash +\sigma^* \text{comp}(r)$. Thus the body of the rule $\text{inf}(r)$ in $T(D)$ holds defeasibly. On the other hand, for every rule $s$ for $\sim q$ in $D$ where $r \not> s$ there is a literal $b$ in the body of $s$ such that $T(D)+A \vdash -\sigma^* \text{supp}(b)$ so, using the inference rule for $-\sigma^*$ and the rule from part 4 we must have $T(D)+A \vdash -\delta^* b$, $T(D)+A \vdash +\delta^* B_s$, so, using the rules in part 4 and part 5, $T(D)+A \vdash +\delta^* \text{supp}_D(r)$. Hence, for the rules for $\sim q$ where $r > s$, the rules $n_d(s,r)$ can be applied and $T(D)+A \vdash -\delta^* \text{comp}(s)$. Consequently, all rules $\text{inf}(s)$ for $\sim q$ fail. From this fact and the fact that body of rule $\text{inf}(r)$ is proved defeasibly we conclude $T(D)+A \vdash +\delta^* q$.

Suppose $-\sigma^* q \in T_{D+A} \uparrow (n+1)$. Then, by the $-\sigma^*$ inference rule, $-\Delta q \in T_{D+A} \uparrow n$ and, for every strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$, either $-\sigma^* b \in T_{D+A} \uparrow n$ for some $b \in B_r$, or there is a rule $s$ in $D$ for $\sim q$ with body $B_s$ such that $+\sigma^* B_s \subseteq T_{D+A} \uparrow n$ and $s > r$. Hence, by the induction hypothesis, $T(D)+A \vdash -\Delta q$ and for every strict or defeasible rule $r$ in $D$ with head $q$ either $T(D)+A \vdash -\delta^* \text{supp}(b)$ for some $b \in B_r$, or there is a rule $s$ in $D$ for $\sim q$ with $s > r$ and $T(D)+A \vdash +\delta^* B_s$. Hence, either $T(D)+A \vdash -\delta^* \text{supp}_D(r)$ or $T(D)+A \vdash -\delta^* -\sigma(r)$. In either case, we have $T(D)+A \vdash -\sigma^* \text{supp}(q)$.

Suppose $-\delta^* q \in T_{D+A} \uparrow (n+1)$. Then, by the $-\delta^*$ inference rule, $-\Delta q \in T_{D+A} \uparrow n$ and, for every strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$, either $-\delta^* b \in T_{D+A} \uparrow n$ for some $b \in B_r$, $+\Delta q \in T_{D+A} \uparrow n$, or there is a rule $s$ in $D$ for $\sim q$ with body $B_s$ such that $+\sigma^* B_s \subseteq T_{D+A} \uparrow n$ and $r \not> s$. Hence, by the induction hypothesis, $T(D)+A \vdash -\Delta q$ and for every strict or defeasible rule $r$ in $D$ with head $q$ either (1) $T(D)+A \vdash -\delta^* b$ for some $b \in B_r$, (2) $T(D)+A \vdash +\Delta q$, or (3) there is a rule $s$ in $D$ for $\sim q$ with $r \not> s$ and $T(D)+A \vdash +\delta^* \text{supp}_D(r)$ for each $b \in B_r$. We consider these three cases in turn. In the first case, the rule $\text{inf}(r)$ fails. In the second case, using part 3, we can conclude $T(D)+A \vdash +\Delta \text{strict}(\sim q)$ and $T(D)+A \vdash -\sigma^* \text{strict}(\sim q)$, and hence the rule $\text{inf}(r)$ fails. In the third case, we can conclude
Suppose $+\vartheta^*\text{supp}(q) \in T_{T(D)+A} \uparrow (n+1)$. Then, by the $+\vartheta^*$ inference rule, either $+\vartheta^*q \in T_{T(D)+A} \uparrow n$, or there is a strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$ such that $+\vartheta^*\text{supp}_b\text{ody}(r) \in T_{T(D)+A} \uparrow n$ and $+\vartheta^*\neg\text{o}(r) \in T_{T(D)+A} \uparrow n$. Consequently, $+\vartheta^*\text{supp}(b) \in T_{T(D)+A} \uparrow n$, for each $b \in B_r$ for every rule $s$ in $D$ for $\neg q$ where $s > r$, there is $b$ in the body of $s$ such that $\neg\vartheta^*b \in T_{T(D)+A} \uparrow n$. In the first case, by the induction hypothesis, $D+A \vdash +\vartheta^*q$ and then, by the inclusion theorem, $D+A \vdash +\sigma^*q$. In the second case, by the induction hypothesis, $D+A \vdash +\sigma^*b$. For every rule $s$ in $D$ for $\neg q$ where $s > r$, there is $b$ in the body of $s$ such that $D+A \vdash -\vartheta^*b$. Applying the inference rule for $+\sigma^*$, $D+A \vdash +\sigma^*q$.

Suppose $-\vartheta^*\text{supp}(q) \in T_{T(D)+A} \uparrow (n+1)$. Then, by the $-\vartheta^*$ inference rule, $-\vartheta^*q \in T_{T(D)+A} \uparrow n$, and for every strict or defeasible rule $r$ in $D$ for $q$ either $-\vartheta^*\text{supp}_b\text{ody}(r) \in T_{T(D)+A} \uparrow n$ or $-\vartheta^*\neg\text{o}(r) \in T_{T(D)+A} \uparrow n$. In the former case we must have $-\vartheta^*\text{supp}(b) \in T_{T(D)+A} \uparrow n$ for some $b$ in the body of $B_r$. In the latter case we must have that for some rule $s$ in $D$ with body $B_s$, $s > r$ and $+\vartheta^*B_s \subseteq T_{T(D)+A} \uparrow n$. By the induction hypothesis, we have $D+A \vdash -\vartheta^*(q)$ (and hence $D+A \vdash -\Delta q$) and, for each $r$ either $D+A \vdash -\sigma^*b$ for some $b \in B_r$, or there is an opposing rule $s$ with $s > r$ and $D+A \vdash +\delta^*B_s$. Applying the inference rule for $-\sigma^*$ we conclude $D+A \vdash -\sigma^*q$.

Suppose $+\vartheta^*q \in T_{T(D)+A} \uparrow (n+1)$. Then, by the $+\vartheta^*$ inference rule, there is a strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$ such that $+\vartheta^*B_r \subseteq T_{T(D)+A} \uparrow n$, $+\vartheta^*\neg\text{strict}(\neg q) \in T_{T(D)+A} \uparrow n$, and $+\vartheta^*\text{comp}(r) \in T_{T(D)+A} \uparrow n$. By Lemma 15, $D+A \vdash -\Delta q$. Using the structure of $T(D)$ and the $+\vartheta^*$ inference rule, for every rule $s$ in $D$ for $\neg q$ where $r \neq s$ we must have $-\vartheta^*\text{supp}_b\text{ody}(s) \in T_{T(D)+A} \uparrow n$, and hence $-\vartheta^*\text{supp}(b) \in T_{T(D)+A} \uparrow n$, for some $b$ in the body of $s$. By the induction hypothesis, $D+A \vdash +\vartheta^*B_r$, and for every rule $s$ in $D$ for $\neg q$ where $r \neq s$, there is $b$ in the body of $s$ such that $D+A \vdash -\sigma^*b$. Now, applying the $+\delta^*$ inference rule, we have $D+A \vdash +\vartheta^*q$.

Suppose $-\vartheta^*q \in T_{T(D)+A} \uparrow (n+1)$. Then, by the $-\vartheta^*$ inference rule, $-\Delta q \in T_{T(D)+A} \uparrow n$ and, for every strict or defeasible rule $r$ for $q$ in $D$ with body $B_r$, either (1) $-\vartheta^*b \in T_{T(D)+A} \uparrow n$ for some $b \in B_r$, (2) for some rule $s'$ in $D$ we have $+\vartheta^*\text{supp}(b) \in T_{T(D)+A} \uparrow n$ for each $b \in B_{s'}$, (3) $+\vartheta^*\text{comp}(\neg q) \in T_{T(D)+A} \uparrow n$, or (4) for some rule $s$ for $\neg q$ in $D$, $+\vartheta^*Q_s \subseteq T_{T(D)+A} \uparrow n$. Consequently, $+\vartheta^*\text{compl}(s) \in T_{T(D)+A} \uparrow n$, and $+\vartheta^*\neg\text{strict}(q) \in T_{T(D)+A} \uparrow n$.

Hence, using the structure of $T(D)$ and Lemma 15, $-\Delta q \in T_{T(D)+A} \uparrow n$ and, for every rule $r$ for $q$ in $D$ with body $B_r$, either (1) $-\vartheta^*b \in T_{T(D)+A} \uparrow n$ for some $b \in B_r$, (2) for some rule $s'$ in $D$ we have $+\vartheta^*\text{supp}(b) \in T_{T(D)+A} \uparrow n$ for each $b \in B_{s'}$, (3) $+\vartheta^*\neg\text{comp}(\neg q) \in T_{T(D)+A} \uparrow n$, or (4) for some rule $s$ for $\neg q$ in $D$, $+\vartheta^*Q_s \subseteq T_{T(D)+A} \uparrow n$. Consequently, $+\vartheta^*\text{compl}(s) \in T_{T(D)+A} \uparrow n$, and $+\vartheta^*\neg\text{strict}(q) \in T_{T(D)+A} \uparrow n$.

By the induction hypothesis, $D+A \vdash -\Delta q$ and, for every strict or defeasible rule $r$ for $q$ in $D$ with body $B_r$, either (1) $D+A \vdash -\delta b$ for some $b \in B_r$, (2) for some rule $s$ for $\neg q$ in $D$ we have $D+A \vdash +\vartheta^*b$ for each $b \in B_r$, (3) $D+A \vdash +\delta^*\neg\text{comp}(\neg q)$, or (4) for some rule $s$ for $\neg q$ in $D$, $D+A \vdash +\delta^*B_s$, for every rule $r'$ for $q$, there is $b'$ in its body such that $D+A \vdash -\sigma^*b'$, and $D+A \vdash -\Delta q$. For each disjunct, applying the inference rule for $-\vartheta^*$, we can conclude $D+A \vdash -\vartheta^*q$.

\[\square\]

This result concerns only addition of facts. It was established in (Maher 2012) that it cannot be extended to addition of rules.
Given that the ambiguity blocking logics can simulate each other, as can the ambiguity propagating logics (see (Maher 2012)) we have

**Theorem 17**
The ambiguity blocking logics (DL(δ) and DL(δ*)) can simulate the ambiguity propagating logics (DL(δ) and DL(δ*)) with respect to addition of facts.

This is Theorem 5 from the body of the paper.

**Propagated Ambiguity Simulates Blocked Ambiguity**

As with the previous simulation, the facts and strict rules of D and T(D) are the same, except for rules for strict(q) in T(D). Thus, again, for any addition A, D+A and T(D)+A draw the same strict conclusions in Σ(D+A). Furthermore, these conclusions are reflected in the defeasible conclusions of strict(q), true(q) and ¬true(q), and also in support conclusions.

**Lemma 18**
Let D be a defeasible theory, T(D) be the transformed defeasible theory as described in Definition 6, and let A be a modular defeasible theory. Let Σ be the language of D+A and let q ∈ Σ. Then

- \( D+A \vdash +\Delta q \) if \( T(D)+A \vdash +\Delta q \) if \( T(D)+A \vdash +\delta startrue(q) \)
  - iff \( T(D)+A \vdash +\delta*true(q) \)
  - iff \( T(D)+A \vdash -\delta*true(q) \) iff \( T(D)+A \vdash -\sigma*true(q) \)

- \( D+A \vdash -\Delta q \) iff \( T(D)+A \vdash -\Delta q \) iff \( T(D)+A \vdash -\delta*true(q) \)
  - iff \( T(D)+A \vdash -\delta*true(q) \)
  - iff \( T(D)+A \vdash +\delta*true(q) \)

**Proof**
The proof of \( D+A \vdash \pm\Delta q \) iff \( T(D)+A \vdash \pm\Delta q \) is straightforward, by induction on length of proofs.

In the inference rule for \(+\delta* true(q)\), clause .2.3 must be false, by the structure of the rules in part 3 of the transformation. Consequently, we infer \(+\delta* true(q)\) iff we infer \(+\Delta true(q)\), which happens iff we infer \( +\Delta q \) since there is only the one rule for \( true(q) \). Similarly, clause .2.3 of the inference rule for \(-\delta* true(q)\) is true, so we infer \(-\delta* true(q)\) iff we infer \( -\Delta true(q) \), which happens iff we infer \( -\Delta q \) since there is only the one rule for \( true(q) \).

Note that \(-\Delta true(q)\) and \(-\Delta false(q)\) are consequences of \( T(D)+A \) because there are no strict rules for such literals in \( T(D)+A \). Using this fact, the two rules \( t(q) \) and \( nt(q) \) and the superiority \( t(q) > nt(q) \), using the inference rule for \(+\delta*\), we can infer \(+\delta*false(q)\) iff we can infer \(-\sigma*false(q)\), because .1 of the inference rule is false, .2.1 and .2.2 are true, and .2.3.2 is false. Similarly, using the inference rule for \(+\sigma*\), we can infer \(+\sigma*false(q)\) iff we can infer \(-\delta*false(q)\). Using the inference rules for \(-\delta* \) and \(-\sigma*\), we can infer \(-\delta*true(q)\) iff we can infer \(+\delta*true(q)\), and we can infer \(-\sigma*true(q)\) iff we can infer \(+\delta*true(q)\).

We need this more detailed characterization of strict consequence, compared to Lemma 15, because both \( \delta* \) and \( \sigma* \) intermediate conclusions influence \( \delta* \) conclusions.

The next lemma is a key part of the proof. It shows that the structure of \( T(D)+A \) tightly constrains the inferences that can be made in the sense that, for the literals of interest, the inference rules \( \delta* \) and \( \sigma* \) draw the same conclusions.
Lemma 19

Let $D$ be a defeasible theory, $T(D)$ be the transformed defeasible theory as described in Definition 6, and let $A$ be a modular set of facts. Let $Σ$ be the language of $D + A$ extended with literals of the forms undefeated($p$), ¬undefeated($p$) and ¬true($p$), for $p ∈ Σ(D)$.

Then, for any $q ∈ Σ$,

- $T(D) + A ⊢ +δ^∗ q$ if and only if $T(D) + A ⊢ +σ^∗ q$
- $T(D) + A ⊢ −δ^∗ q$ if and only if $T(D) + A ⊢ −σ^∗ q$

Proof

Two parts of the proof follow immediately from the inclusion theorem. These are the forward direction of the first statement and the backward direction of the second statement. Furthermore, it is immediate from Lemma 18 that the result holds for literals involving true and for literals that are proved strictly. The remaining parts are proved by induction.

Recall that $T(D) + A ⊢ s$ if there is an integer $n$ such that $s ∈ T_{T(D)+A} ⊢ n$. Note that the result holds in $T_{T(D)+A} ⊢ 0$, since it is empty. Suppose the result holds for conclusions $s$ with $s ∈ T_{T(D)+A} ⊢ n$. We show that it also holds for conclusions in $T_{T(D)+A} ⊢ (n + 1)$.

1. If $+σ^∗ q ∈ T_{T(D)+A} ⊢ (n + 1)$ then $+σ^∗ defeated(q) ∈ T_{T(D)+A} ⊢ n$, because there is only one rule for $q$ and it cannot be overruled. Further, if $+σ^∗ defeated(q) ∈ T_{T(D)+A} ⊢ n$ then for some rule $r$ of $D$ we must have $+σ^∗ B_r ⊆ T_{T(D)+A} ⊢ n$ and $+σ^∗ ¬true(¬q) ∈ T_{T(D)+A} ⊢ n$ and, for every rule $s$ of $D$ for $¬q$ with $r$, there is $p ∈ B_r$ with $−δ^∗ p ∈ T_{T(D)+A} ⊢ n$, because clause 2.2.2 must be false, since $n_{d(r, s)} > p_d(r)$ for every such $s$.

By the induction hypothesis, $+δ^∗ B_r ⊆ T_{T(D)+A} ⊢ n$, and for each $s$ there is $p ∈ B_s$ with $−σ^∗ p ∈ T_{T(D)+A} ⊢ n$ and, by Lemma 18, $+δ^∗ ¬true(¬q)$ and $−Δq ⊢ T(D)+A$. Applying the $+δ^∗$ inference rule, $T(D)+A ⊢ +δ^∗ defeated(q)$ and, applying the $−σ^∗$ inference rule, $T(D)+A ⊢ −σ^∗ defeated(¬q)$ since every rule $p_d(s)$ contains a $p$ with $−σ^∗ p ∈ T_{T(D)+A} ⊢ n$. Hence, applying the $+δ^∗$ inference rule, $T(D)+A ⊢ +δ^∗ q$.

If $+σ^∗ defeated(q) ∈ T_{T(D)+A} ⊢ (n + 1)$ then for some rule $s$ of $D$, $+σ^∗ B_s ⊆ T_{T(D)+A} ⊢ n$. By the induction hypothesis, $+δ^∗ B_s ⊆ T_{T(D)+A} ⊢ n$. Applying the $+δ^∗$ inference rule, noting that there is no fact or strict rule for defeated($q$) and that $n_{d(r, s)} > p_d(r)$, we have $T(D)+A ⊢ +δ^∗ defeated(q)$.

1. If $−δ^∗ q ∈ T_{T(D)+A} ⊢ (n + 1)$ then, using the $−δ^∗$ inference rule and the structure of $T(D)+A$, $−Δq ∈ T_{T(D)+A} ⊢ n$ and either $−δ^∗ defeated(q) ∈ T_{T(D)+A} ⊢ n$ or $−Δq ∈ T_{T(D)+A} ⊢ n$ or $+σ^∗ defeated(¬q) ∈ T_{T(D)+A} ⊢ n$.

If $−δ^∗ defeated(q) ∈ T_{T(D)+A} ⊢ (n + 1)$ then either for every rule $p_d(r)$, for some $p ∈ B_r$, $−δ^∗ p ∈ T_{T(D)+A} ⊢ n$ or $−δ^∗ true(¬q) ∈ T_{T(D)+A} ⊢ n$, or, for some rule $n_{d(r, s)}$, $+σ^∗ B_s$. By the induction hypothesis, either for each $p_d(r)$ there is a $p$ in its body where $T(D)+A ⊢ −σ^∗ p$, or $T(D)+A ⊢ +δ^∗ B_s$ for some $s$. Applying the $−σ^∗$, we have $T(D)+A ⊢ −σ^∗ defeated(q)$.

If $−Δq ∈ T_{T(D)+A} ⊢ n$ then, by Lemma 18, $T(D)+A ⊢ −σ^∗ true(¬q)$. Hence we must have $T(D)+A ⊢ −σ^∗ defeated(q)$, since $¬true(¬q)$ appears in each rule for defeated($q$).

If $+σ^∗ defeated(¬q) ∈ T_{T(D)+A} ⊢ n$ then there is a rule $p_d(s)$ for defeated($¬q$) where $T(D)+A ⊢ +σ^∗ B_s$ and $T(D)+A ⊢ +σ^∗ true(q)$ and, for every rule $n_{d(s, r)}$, $T(D)+A ⊢ ¬δ^∗ B_r$. By the induction hypothesis, $T(D)+A ⊢ +δ^∗ B_s$ and, for every rule $n_{d(s, r)}$ (where we must have $s_r r$ in $D$), $T(D)+A ⊢ −δ^∗ B_r$. Hence, for every rule $q$ of $D$ where $r > s$ we have $T(D)+A ⊢ −σ^∗ B_r$. For every other $r$ for $q$ in $D$ there is $n_{d(r, s)}$ where $T(D)+A ⊢ +δ^∗ B_r$. Hence, applying the $−σ^∗$ inference rule for defeated($q$), we must have $T(D)+A ⊢ −σ^∗ defeated(q)$. 

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Thus, in every case we have $T(D)+A \vdash \neg \sigma^* \text{undefeated}(q)$ and consequently $T(D)+A \vdash \neg \sigma^* q$.

If $-\sigma^* \text{undefeated}(q) \in T_{T(D)+A} \uparrow (n+1)$ then for every rule $p_d(r)$, for some $p$ in its body, $-\sigma^* p \in T_{T(D)+A} \uparrow n$. By the induction hypothesis, for every rule $p_d(r)$, for some $p$ in its body, $T(D)+A \vdash -\sigma^* p$. Applying the $-\sigma^*$ inference rule, $T(D)+A \vdash -\sigma^* -\text{undefeated}(q)$.

As a consequence of the inclusion theorem and the previous lemma, any inference rule between $\sigma^*$ and $\delta^*$ (that is, any inference rule except for $\Delta$ and $\delta$) behaves the same way on $\Sigma$-literals in $T(D)+A$. In particular, it applies to $\delta^*$.

**Corollary 20**

Let $\Sigma$ be the language of $D$, $\Sigma'$ be as defined in the previous lemma. Let $A$ be any set of facts. Then if $q \in \Sigma'$

- $T(D)+A \vdash +\delta^* q$ iff $T(D)+A \vdash +\partial^* q$
- $T(D)+A \vdash -\delta^* q$ iff $T(D)+A \vdash -\partial^* q$

Now we show that the transformation preserves the $\partial^*$ consequences of $D+A$.

**Theorem 21**

Let $D$ be a defeasible theory, $T(D)$ be the transformed defeasible theory as described in Definition 6, and let $A$ be a modular set of facts. Let $\Sigma$ be the language of $D+A$ and let $q \in \Sigma$. Then

- $D+A \vdash +\partial^* q$ iff $T(D)+A \vdash +\partial^* q$
- $D+A \vdash -\partial^* q$ iff $T(D)+A \vdash -\partial^* q$

**Proof**

Suppose $+\partial^* q \in T_{D+A} \uparrow (n+1)$. Then, by the $+\partial^*$ inference rule, either $+\Delta q \in T_{D+A} \uparrow n$ (in which case, we must have $T(D)+A \vdash +\partial^*$) or $+\Delta q \notin T_{D+A} \uparrow n$ and there is a strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$ such that $+\partial^* B_r \subseteq T_{D+A} \uparrow n$, $-\Delta q \in T_{D+A} \uparrow n$, and for every rule $s$ in $D$ for $\sim q$ either there is a literal $b$ in the body of $s$ such that $-\partial^* b \in T_{D+A} \uparrow n$ or $r > s$. Hence, in the latter case, by the induction hypothesis, there is a strict or defeasible rule $r$ in $D+A$ with head $q$ and body $B_r$ such that $T(D)+A \vdash +\partial^* B_r$, $T(D)+A \vdash -\Delta \sim q$, and for every rule $s$ in $D+A$ for $\sim q$ either $T(D)+A \vdash -\partial^* B_s$ or $r > s$.

From this statement we derive several facts. (1) By Lemma 18 and the inclusion theorem, $T(D)+A \vdash +\partial^* \text{true} \sim q)$. (2) Thus, $T(D)+A \vdash +\partial^* (B_r, \sim \text{true} \sim q))$ and, for every rule $n_d(s, r)$ in $T(D)$, $T(D)+A \vdash -\partial^* B_s$ (since rules $s$ where $r > s$ do not give rise to a rule $n_d(r, s)$). Hence, $T(D)+A \vdash +\partial^* \text{undefeated}(q)$. (3) Conversely, $T(D)+A \vdash -\partial^* \text{undefeated}(q)$ because, for every rule $p_d(s)$ for $\text{undefeated}(q)$, either $T(D)+A \vdash -\partial^* B_s$ or there is a rule $n_d(s, r)$ superior to $p_d(s)$ with $T(D)+A \vdash +\partial^* B_r$. Consequently, since the only rule in $T(D)$ for $q$ has body $\text{undefeated}(q)$ (and similarly for $\sim q$), applying the $+\partial^*$ inference rule, we have $T(D)+A \vdash +\partial^* q$.

Suppose $-\partial^* q \in T_{D+A} \uparrow (n+1)$. Then, by the $-\partial^*$ inference rule, $-\Delta q \in T_{D+A} \uparrow n$ and, for every strict or defeasible rule $r$ in $D$ with head $q$ and body $B_r$, either $-\partial^* b \subseteq T_{D+A} \uparrow n$ for some $b \in B_r$, $+\Delta \sim q \in T_{D+A} \uparrow n$, or there is a rule $s$ in $D$ for $\sim q$ with body $B_s$ such that $+\partial^* B_s \subseteq T_{D+A} \uparrow n$ and $r \neq s$. Hence, by the induction hypothesis, $T(D)+A \vdash -\Delta q$ and for every strict or defeasible rule $r$ in $D$ with head $q$ either $T(D)+A \vdash -\partial^* b$ for some
b ∈ B_r, T(D)+A ⊢ +Δ~q, or there is a rule s in D for ~q where T(D)+A ⊢ +δ^* B_s and r ≠ s. Hence, for every rule p_d(r) in T(D) for undefeated(q) either T(D)+A ⊢ −δ^* b for some b ∈ B_r, or T(D)+A ⊢ −δ^* ~strict(~q) (by Lemma 18), or there is a rule n_d(r,s) where T(D)+A ⊢ +σ^* B_s. Applying the inference rule for −δ^* undefeated(q), we conclude T(D)+A ⊢ −δ^* q and, hence, T(D)+A ⊢ −δ^* q.

Suppose +δ^* q ∈ T_{T(D)+A} ↑ (n+1). Then, by the +δ^* inference rule and using the structure of T(D), either +Δq ∈ T_{T(D)+A} ↑ n, or +δ^* undefeated(q) ∈ T_{T(D)+A} ↑ n, −Δ~q ∈ T_{T(D)+A} ↑ n, and −δ^* undefeated(~q) ∈ T_{D+A} ↑ n. In the first case we have D+A ⊢ +Δq and thus D+A ⊢ +δ^* q. Alternatively, there is a strict or defeasible rule r in D with head q and body B_r such that +δ^* B_r ⊆ T_{T(D)+A} ↑ n, +δ^* ~true(~q) ∈ T_{T(D)+A} ↑ n, and for every rule s in D for ~q where r ≠ s there is a literal b in the body B_s of s such that −δ^* b ∈ T_{T(D)+A} ↑ n. By the induction hypothesis and Lemma 18, D+A ⊢ +δ^* B_r, D+A ⊢ −Δ~q, and for every rule s in D for ~q where r ≠ s there is b in the body of s such that D+A ⊢ −δ b. Applying the +δ inference rule, we conclude D+A ⊢ +δ^* q.

Suppose −δ^* q ∈ T_{T(D)+A} ↑ (n+1). Then, by the −δ^* inference rule and using the structure of T(D), either −Δq ∈ T_{T(D)+A} ↑ n and either (1) −δ^* undefeated(q) ∈ T_{T(D)+A} ↑ n, or (2) +Δ~q ∈ T_{T(D)+A} ↑ n, or (3) −δ^* undefeated(~q) ∈ T_{T(D)+A} ↑ n. By Lemma 18 and Corollary 20 we have D+A ⊢ −Δq and D+A ⊢ +δ^* ~true(q).

In the first case, for each rule r for q in D either there is a literal p ∈ B_r, and −δ p ∈ T_{T(D)+A} ↑ n or −δ^* ~true(q) ∈ T_{T(D)+A} ↑ n or for some rule s for ~q in D where r ≠ s, +δ^* B_s ⊆ T_{T(D)+A} ↑ n. By the induction hypothesis (and Lemma 18 and Corollary 20), for each rule r for q in D either there is a literal p ∈ B_r, and D+A ⊢ −δ^* p, or D+A ⊢ +Δ~q, or for some rule s for ~q in D where r ≠ s, D+A ⊢ +δ^* B_s. Applying the −δ inference rule, D+A ⊢ −δ^* q.

In the second case, by Lemma 18 and Corollary 20, D+A ⊢ +Δ~q. Consequently, applying the −δ inference rule, D+A ⊢ −δ^* q. In the third case, for some rule s for ~q in D, +δ^* B_s ⊆ T_{T(D)+A} ↑ n and, for all rules r for q in D where s ≠ r, for some p ∈ B_r, −δ^* p ∈ T_{T(D)+A} ↑ n. By the induction hypothesis, for every rule r for q in D where r > s, for some p ∈ B_r, D+A ⊢ −δ^* p, and D+A ⊢ +δ^* B_s. Applying the −δ inference rule, D+A ⊢ −δ^* q.

□

Combining Theorem 21 with Lemma 19 and the inclusion theorem, we see that DL(δ^*) can be simulated by DL(δ^*) and DL(δ).

Theorem 22
For d ∈ {δ, δ^*, δ}, DL(d) can simulate DL(δ^*) with respect to addition of facts

Proof
D+A ⊢ +δ^* q iff T(D)+A ⊢ +δ^* q (by Theorem 21) iff T(D)+A ⊢ +δ q (by Corollary 20) iff T(D)+A ⊢ +δ q (by Lemma 19 and the inclusion theorem). The proof is similar for −δ^* q.

□

This is Theorem 8 from the body of the paper.

Simulation of Individual Defeat wrt Addition of Rules

Example 10 does not apply to DL(δ) and DL(δ^*). We have T(D)+A ⊢ +σh(r_2) and, consequently, T(D)+A ⊢ −δ p, in agreement with D under DL(δ^*). The weaker inference strength
of ambiguity propagation masks the distinction that is present for blocked ambiguity reasoning. However, the next example shows that the transformation does not provide a simulation wrt rules for the propagating ambiguity logics.

**Example 23**
Let $D$ consist of the rules

- $r_1 : \Rightarrow p$
- $r_2 : \Rightarrow \neg p$
- $r_3 : \Rightarrow p$
- $r_4 : \Rightarrow \neg p$

with $r_1 > r_2$ and $r_3 > r_4$.

Then $\mathcal{T}(D)$ consists of the following rules

- $p(r_1) : \Rightarrow h(r_1)$
- $n(r_1, r_2) : \Rightarrow \neg h(r_1)$
- $n(r_1, r_4) : \Rightarrow \neg h(r_1)$
- $p(r_2) : \Rightarrow h(r_2)$
- $n(r_2, r_1) : \Rightarrow \neg h(r_2)$
- $n(r_2, r_3) : \Rightarrow \neg h(r_2)$
- $p(r_3) : \Rightarrow h(r_1)$
- $n(r_3, r_2) : \Rightarrow \neg h(r_3)$
- $n(r_3, r_4) : \Rightarrow \neg h(r_3)$
- $p(r_4) : \Rightarrow h(r_4)$
- $n(r_4, r_1) : \Rightarrow \neg h(r_4)$
- $n(r_4, r_3) : \Rightarrow \neg h(r_4)$

with $p(r_1) > n(r_1, r_2), n(r_2, r_1) > p(r_2), p(r_3) > n(r_3, r_4)$, and $n(r_4, r_3) > p(r_4)$.

Now, let $A$ be the rule

- $\Rightarrow p$

Then $D + A \vdash -\delta^* p$, because for every rule $r$ for $p$, there is a rule for $\neg p$ that is not overruled by $r$ ($r_1$ does not overrule $r_4$, $r_3$ does not overrule $r_2$ and $A$ overrules neither).

However, considering the transformed theory, $\mathcal{T}(D) + A \vdash -\sigma h(r_2)$, because $n(r_2, r_1) > p(r_2)$ and, similarly, $\mathcal{T}(D) + A \vdash -\sigma h(r_4)$. Consequently, both rules for $\neg p$ in $\mathcal{T}(D) + A$ fail. This leaves the rules for $p$ without competition, and so $\mathcal{T}(D) + A \vdash +\delta p$, conflicting with the behaviour of $D + A$.

Following essentially the same argument, this example also applies to $\text{DL}(\delta^*)$ and $\text{DL}(\partial)$.

We show that the transformation defined in Definition 11 (and Definition 9 ) allows the team-defeat logics to simulate their individual-defeat counterparts. We treat the two cases separately, but first we address the effect of the transformation on strict inference.

**Lemma 24**
Consider the transformation $T$ from Definition 11. For any $D$ and $A$

- $D + A \vdash +\Delta q$ iff $\mathcal{T}(D) + A \vdash +\Delta q$
• $D + A \vdash -\Delta q$ iff $T(D) + A \vdash -\Delta q$

The proof is a straightforward induction.

**Theorem 25**
The logic $\mathbf{DL}(\partial^*)$ can be simulated by $\mathbf{DL}(\partial)$ with respect to addition of rules.

**Proof**
We consider the transformation $T(D)$ of a defeasible theory $D$ as defined in Definition 11 (and Definition 9) and show that this transformation provides a simulation of each defeasible theory $D$ in $\mathbf{DL}(\partial^*)$ from within $\mathbf{DL}(\partial)$.

Fix any $D$ and any $A$ that satisfies the language separation condition. Let $\Sigma = \Sigma(D) \cup \Sigma(A)$.

If $+\partial^* q \in T_{D+A}\uparrow(n+1)$ then either $+\Delta q \in T_{D+A}\uparrow n$ (in which case $T(D) + A \vdash +\partial q$) or $+\partial^* B_r \subseteq T_{D+A}\uparrow n$, where $B_r$ is the body of some strict or defeasible rule $r$ in $D+A$. In the latter case, $T_{D+A}\uparrow n$ also contains $-\Delta \sim q$ and for every rule $s$ for $\sim q$ in $D+A$ either $s > r$ or $-\partial^* \in T_{D+A}\uparrow n$ for some literal $p$ in the body of $s$. Then, by the induction hypothesis, $T(D) + A \vdash +\partial B_r$. $T(D) + A \vdash -\Delta \sim q$ and, if $r \in D$, for every rule $n(r,s)$ for $-h(r)$ in $T(D)$, either $p(r) >' n(r,s)$ or $T(D) + A \vdash -\partial p$ where $p$ occurs in the body of $n(r,s)$. Thus, using the inference rule for $+\partial$ if $r \in D$ then $T(D) + A \vdash +\partial h(r)$. If $r \in A$ then $T(D) + A \vdash +\partial B_r$, so, whether $r \in D$ or $r \in A$, there is a rule for $q$ in $T(D) + A$ with body $B$ and $T(D) + A \vdash +\partial B_r$.

Applying the inference rule for $-\partial$ multiple times, for each strict or defeasible rule $s$ for $\sim q$ in $D$ we have $T(D) + A \vdash -\partial h(s)$. Furthermore, as noted above, for every rule $s \in A$ for $\sim q$, since $r \not> s$, $-\partial^* p \in T_{D+A}\uparrow n$ for some literal $p$ in the body of $s$. Thus, every rule for $\sim q$ in $T(D) + A$ fails. Now, again applying the inference rule for $+\partial$, we have $T(D) + A \vdash +\partial^* q$.

If $-\partial^* q \in T_{D+A}\uparrow(n+1)$ then $-\Delta q \in T_{D+A}\uparrow n$ and either $+\Delta \sim q \in T_{D+A}\uparrow n$ (in which case $T(D) + A \vdash -\partial q$) or, for every strict or defeasible rule $r$ for $q$ in $D+A$, either $-\partial^* p \in T_{D+A}\uparrow n$ for some $p$ in the body of $r$ or there exists a rule $s$ for $\sim q$ with body $B$, $+\partial^* B_r \subseteq T_{D+A}\uparrow n$ and $r \not> s$. Then, for every strict or defeasible rule $p(r)$ in $T(D)$, either $-\partial^* p \in T_{D+A}\uparrow n$ for some $p$ in the body of $p(r)$ or there is a rule $n(r,s)$ with body $B$ and $p(r) >' n(r,s)$, by the structure of $T(D)$. By the induction hypothesis, for every strict or defeasible rule $p(r)$ in $T(D)$, either $T(D) + A \vdash -\partial p$ for some $p$ in the body of $p(r)$ or there is a rule $n(r,s')$ with body $B$ where $T(D) + A \vdash +\partial B$ and $p(r) >' n(r,s')$. Since there is only one rule for $h(r)$, application of the inference rule for $-\partial$ gives us $T(D) + A \vdash -\partial h(r)$ for each strict or defeasible rule $r \in D$ for $q$. Also by the induction hypothesis, for every strict or defeasible rule $r$ for $q$ in $A$, $T(D) + A \vdash -\partial q$ for some $p$ in the body of $r$. Hence $T(D) + A \vdash -\partial q$.

If $q \in \Sigma$ and $+\partial q \in T_{T(D)+A}\uparrow(n+1)$ then either (1) $+\Delta q \in T_{T(D)+A}\uparrow n$ (in which case $D+A \vdash +\partial^* q$), or else (2) $+\partial h(r) \in T_{T(D)+A}\uparrow n$ for some strict or defeasible rule $r$ for $q$ in $D$, or else (3) $+\partial B_r \subseteq T_{T(D)+A}\uparrow n$ for some strict or defeasible rule $r$ for $q$ in $A$ with body $B_r$. In case (3), by the induction hypothesis, $D+A \vdash +\partial B_r$. In case (2) we must also have that every rule for $\sim q$ in $T(D)+A$ fails (except for $o(\sim q)$, which is overruled); that is, for every rule $s$ for $\sim q$ in $D$, $-\partial h(s) \in T_{T(D)+A}\uparrow n$ and, for every rule for $\sim q$ in $A$ with body $B$, for some literal $p$ in $B -\partial p \in T_{T(D)+A}\uparrow n$. In case (3) we must also have that $\texttt{one}(\sim q)$ fails, so that every rule for $\sim q$ in $D$ with body $B$, for some literal $p$ in $B -\partial p \in T_{T(D)+A}\uparrow n$. Hence, by the induction hypothesis, in case (3), for every rule for $\sim q$ in $D+A$ with body $B$, for some literal $p$ in $B D+A \vdash -\partial p$. In both cases (2) and (3), $-\Delta q \in T_{T(D)+A}\uparrow n$ and hence $D+A \vdash -\Delta \sim q$. Applying the $+\partial^*$ inference rule in case (3), $D+A \vdash +\partial^* q$.

In case (2), if $+\partial h(r) \in T_{T(D)+A}\uparrow n$ then $p(r)$ is not a defeater, $+\partial B_r \subseteq T_{T(D)+A}\uparrow n$ where
Let \( B_r \) be the body of \( r \) and for every rule \( n(r, s) \) with body \( B' \) either for some literal \( p \) in \( B' - \partial p \in T_{T(D)+A} \nabla n \) or for some rule \( t \) for \( h(r) \), its body is proved with respect to \( \partial \) and \( t > s \). There is only one rule for \( h(r) \), so this last disjunct reduces to \( p(r) > n(r, s) \). Using the construction of \( T(D), r \) is not a defeater, \( +\delta B_r \subseteq T_{T(D)+A} \nabla n \) where \( B_r \) is the body of \( r \) and for every rule \( s \) for \( \sim q \) in \( D \) with body \( B' \) either for some literal \( p \) in \( B' - \partial p \in T_{T(D)+A} \nabla n \) or \( r > s \). Furthermore, from the previous paragraph, for every rule for \( \sim q \) in \( A \) with body \( B \), for some literal \( p \) in \( B - \partial p \in T_{T(D)+A} \nabla n \). Using the induction hypothesis, \( D + A \vdash +\sigma^* B_r \), and for every rule for \( \sim q \) in \( D + A \) either for some literal \( p \) in the body \( D + A \vdash -\delta^* p \) or \( r > s \). Applying the inference rule for \( +\sigma^* \), we obtain \( D + A \vdash -\delta^* q \).

[4] If \( q \in \Sigma \) and \( -\partial q \in T_{T(D)+A} \nabla (n+1) \) then, using the \( -\sigma^* \) inference rule and the structure of \( T(D), -\Delta q \in T_{T(D)+A} \nabla n \) and either (0) \( +\Delta \sim q \in T_{T(D)+A} \nabla n \) (in which case \( D + A \vdash -\sigma^* q \)), or else (1) \( -\partial h(r) \in T_{T(D)+A} \nabla n \) for every rule \( r \) for \( q \) in \( D \), while for every rule \( r \) in \( A \) there is a literal \( p \) in the body of \( r \) with \( -\partial p \in T_{T(D)+A} \nabla n \), and \( -\partial q \in T_{T(D)+A} \nabla n \); or (2) \( +\partial h(s) \in T_{T(D)+A} \nabla n \) for some rule \( s \) for \( \sim q \) in \( D \); or (3) \( +\partial q \in T_{T(D)+A} \nabla n \), in which case there is a rule \( s \) for \( \sim q \) in \( D \) where \( +\partial B_s \subseteq T_{T(D)+A} \nabla n \), and \( -\partial h(r) \in T_{T(D)+A} \nabla n \) for every rule \( r \) for \( q \) in \( D \) (so that \( o(\sim q) \) is not overruled); or (4) there is a rule \( s \) for \( \sim q \) in \( A \) where \( +\partial B_s \subseteq T_{T(D)+A} \nabla n \). In any case, using the induction hypothesis, we have \( D + A \vdash -\Delta q \).

In case (1), since \( -\partial q \in T_{T(D)+A} \nabla n \), for every rule \( r \) in \( D \) for \( q \) there is a literal \( p \) in \( B_r \) with \( -\partial p \in T_{T(D)+A} \nabla n \). Thus all rules for \( q \) in \( D + A \) fail, and hence \( D + A \vdash -\sigma^* q \). In case (2), we must have, for every rule \( r \) in \( D \) for \( q \), either \( s > r \) (so that \( p(s) > n(s, r) \) or there is a literal \( p \) in \( B_r \) with \( -\partial p \in T_{T(D)+A} \nabla n \). By the induction hypothesis, we then have \( D + A \vdash -\sigma^* p \), for each such \( p \). Furthermore, no rule in \( A \) can overrule \( s \). Hence, applying the \( -\sigma^* \) inference rule, \( D + A \vdash -\sigma^* q \).

In case (3), since \( -\partial h(r) \in T_{T(D)+A} \nabla n \), either there is a literal \( p \) in \( B_r \) with \( -\partial p \in T_{T(D)+A} \nabla n \) or there is a rule \( s \) in \( D \) for \( \sim q \) with \( +\partial B_s \subseteq T_{T(D)+A} \nabla n \) and \( r \not> s \) (so that \( p(r) \not> n(s, r) \)). By the induction hypothesis, for every rule \( r \) for \( q \) in \( D \) either there is a literal \( p \) in \( B_r \) with \( D + A \vdash -\sigma^* p \) or there is a rule \( s \) in \( D \) for \( \sim q \) with \( D + A \vdash -\sigma^* B_s \) and \( r \not> s \). Furthermore, from \( +\partial q \) we know there is a \( s \) in \( D \) with (using the induction hypothesis) \( D + A \vdash -\sigma^* B_s \), and this \( s \) cannot be overruled by any rule \( r \) in \( A \). Consequently, applying the \( -\sigma^* \) inference rule, \( D + A \vdash -\sigma^* q \).

In case (4), by the induction hypothesis, we have there is a rule \( s \) for \( \sim q \) in \( A \) where \( D + A \vdash -\partial q \), and, since \( s \) cannot be inferior to any rule, applying the \( -\sigma^* \) inference rule we have \( D + A \vdash -\sigma^* q \).

This concludes the proof that \( \text{DL}(\delta) \) can simulate \( \text{DL}(\delta^*) \) with respect to addition of rules. We now turn to the corresponding proof for \( \text{DL}(\delta) \) and \( \text{DL}(\delta^*) \).

**Theorem 26**

The logic \( \text{DL}(\delta^*) \) can be simulated by \( \text{DL}(\delta) \) with respect to addition of rules.

**Proof**

Let \( A \) be any set of rules. Let \( \Sigma \) be the language of \( D + A \) and let \( q \in \Sigma \). Let \( T(D) \) be the transformed defeasible theory as described in Definition 11. Then we claim

- \( D + A \vdash +\sigma^* q \) if \( T(D)+A \vdash +\sigma q \)
- \( D + A \vdash -\sigma^* q \) if \( T(D)+A \vdash -\sigma q \)
- \( D + A \vdash +\delta^* q \) if \( T(D)+A \vdash +\delta q \)
\[ D+A \vdash -\delta q \text{ if } T(D) + A \vdash -\delta q \]

If \( +\delta q \in T_{D+A} \uparrow (n+1) \) then either \( +\Delta q \in T_{D+A} \uparrow n \) (in which case \( T(D) + A \vdash +\delta q \)) or else \( +\delta^* B_r \subseteq T_{D+A} \uparrow n \), where \( B_r \) is the body of some strict or defeasible rule \( r \) in \( D+A \). In the latter case, \( T_{D+A} \uparrow n \) also contains \( -\Delta \sim q \) and for every rule \( s \) for \( \sim q \) in \( D+A \) either \( r > s \) or \( -\sigma^* p \in T_{D+A} \uparrow n \) for some literal \( p \) in the body of \( s \). Then, by the induction hypothesis, \( T(D) + A + \delta B_r, T(D) + A \vdash -\Delta \sim q \) and, if \( r \) and \( s \) are in \( D \), for every rule \( n(r,s) \) for \( \sim h(r) \) in \( T(D) \), either \( p(r) >' n(r,s) \) or \( T(D) + A \vdash -\sigma p \) where \( p \) occurs in the body of \( n(r,s) \) and, similarly, the rule \( p(s) \) for \( h(s) \) in \( T(D) \), either \( p(s) <' n(s,r) \) or \( T(D) + A \vdash -\sigma p \) where \( p \) occurs in the body of \( p(s) \). If \( s \) is in \( A \) then \( r > s \) cannot occur (since the rules of \( A \) do not participate in the superiority relation) and \( T(D) + A \vdash -\sigma p \) where \( p \) occurs in the body of \( s \). If \( r \) is in \( A \) and \( s \) is in \( D \) then, again, \( r > s \) cannot occur and \( T(D) + A \vdash -\sigma p \) where \( p \) occurs in the body of \( p(s) \). Thus, using the inference rules for \( +\delta \) and \( -\sigma \), if \( r \) is in \( D \) then \( T(D) + A \vdash +\delta h(r) \) and if \( s \) is in \( D \) then \( T(D) + A \vdash -\sigma h(s) \). Now, applying the inference rule for \( +\delta \), we conclude \( T(D) + A \vdash +\delta q \).

If \( -\delta q \in T_{D+A} \uparrow (n+1) \) then \( -\Delta q \in T_{D+A} \uparrow n \) (and, hence, \( T(D) + A \vdash -\Delta q \)) and either \( +\Delta q \in T_{D+A} \uparrow n \) (in which case \( T(D) + A \vdash -\delta q \)) or, for every strict or defeasible rule \( r \) for \( q \) in \( D+A \), either \( -\delta^* p \in T_{D+A} \uparrow n \) for some \( p \) in the body of \( r \) or there exists a rule \( s \) for \( \sim q \) with body \( B_s \), where \( +\sigma^* B_s \subseteq T_{D+A} \uparrow n \) and \( r \not> s \). Now, if, for some \( s \) for \( \sim q \) in \( A \), \( +\sigma^* B_s \subseteq T_{D+A} \uparrow n \) then, by the induction hypothesis, \( T(D) + A \vdash +\sigma B_s \) and, applying the inference rule for \( -\delta \) (and noting that no rule is superior to \( s \)), we have \( T(D) + A \vdash -\delta q \).

Otherwise, for every strict or defeasible rule \( p(r) \) in \( T(D) \), either \( -\delta^* p \in T_{D+A} \uparrow n \) for some \( p \) in the body of \( p(r) \) or there is a rule \( n(r,s) \) with body \( B_s \) and \( p(r) \not> n(r,s) \), by the structure of \( T(D) \). By the induction hypothesis, for every strict or defeasible rule \( p(r) \) in \( T(D) \), either \( T(D) + A \vdash -\delta p \) for some \( p \) in the body of \( p(r) \) or there is a rule \( n(r,s) \) with body \( B_s \) where \( T(D) + A \vdash +\sigma B_s \) and \( p(r) \not> n(r,s) \). In both cases, since there is only one rule for \( h(r) \), application of the inference rule for \( -\delta \) gives us \( T(D) + A \vdash +\delta h(r) \) for each strict or defeasible rule \( r \) for \( q \) in \( D \). Hence, no rule \( s(r) \) can overrule \( o(\sim q) \). Now, if every rule \( r \) for \( q \) in \( A \) has \( p \in B_r \) with \( -\delta^* p \in T_{D+A} \uparrow n \) then, by the induction hypothesis, \( T(D) + A \vdash -\delta p \) for every such rule \( r \) hence, by application of the \( -\delta \) inference rule, \( T(D) + A \vdash -\delta q \). Otherwise, there is \( s \) for \( \sim q \) in \( D \) with \( T(D) + A \vdash +\sigma B_s \). By the \( +\sigma \) inference rule \( T(D) + A \vdash +\sigma o(\sim q) \). Consequently, since \( o(\sim q) \) cannot be overruled, \( T(D) + A \vdash -\delta q \).

Hence, in every case, \( T(D) + A \vdash -\delta q \).

If \( q \in \Sigma \) and \( +\delta q \in T_{T(D)+A} \uparrow (n+1) \) then either \( +\Delta q \in T_{T(D)+A} \uparrow n \) (in which case \( D+A \vdash +\delta q \), or \(+\delta h(r) \in T_{T(D)+A} \uparrow n \) for some strict or defeasible rule \( r \) for \( q \) in \( D \). or \(+\delta B_r \subseteq T_{T(D)+A} \uparrow n \) for some strict or defeasible rule \( r \) for \( q \) in \( A \) with body \( B_r \). In the latter cases, \( T_{T(D)+A} \uparrow n \) also contains \( -\Delta q \); hence \( D+A \vdash -\Delta q \). In these cases we must also have, for each rule for \( \sim q \) in \( A \), for some \( p \) in its body \( -\sigma p \in T_{T(D)+A} \uparrow n \). Hence, by the induction hypothesis, for each rule for \( \sim q \) in \( A \), for some \( p \) in its body \( D+A \vdash -\sigma^* p \). If \(+\delta h(r) \in T_{T(D)+A} \uparrow n \) then \( p(r) \) is not a defeater, \(+\delta B_r \subseteq T_{T(D)+A} \uparrow n \) where \( B_r \) is the body of \( r \) and for every rule \( n(r,s) \) with body \( B_s \) either for some literal \( p \) in \( B_s \), \( -\sigma p \in T_{T(D)+A} \uparrow n \) or for some rule \( t \), its body is proved with respect to \( \delta \) and \( t > s \). There is only one rule for \( h(r) \), so this last disjunct reduces to \( p(r) > n(r,s) \). Using the construction of \( T(D) \), \( r \) is not a defeater, \(+\delta B_r \subseteq T_{T(D)+A} \uparrow n \) and for every rule \( s \) for \( \sim q \) in \( D \) either for some literal \( p \) in \( B_s \), \( -\sigma p \in T_{T(D)+A} \uparrow n \) or \( t > s \). Using the induction hypothesis, \( D+A \vdash +\delta^* B_r \), and for every rule for \( \sim q \) in \( D \) either for some literal \( p \) in the body \( D+A \vdash -\sigma^* p \) or \( r > s \). Applying
the inference rule for $+\delta^*$ to this statement, and given we have shown that all rules for $\neg q$ in $A$ fail, we obtain $D+A \vdash +\delta^*q$.

If $q \in \Sigma$ and $-\delta q \in \mathcal{T}_{T(D)+A}^+(n+1)$ then $-\Delta q \in \mathcal{T}_{T(D)+A}^+n$ and either (1) $+\Delta q \in \mathcal{T}_{T(D)+A}^+n$ (in which case $D+A \vdash -\delta^*q$), or (2) $-\delta h(r) \in \mathcal{T}_{T(D)+A}^+n$ for every rule $r$ for $q$ in $D$ and every rule for $q$ in $A$ has a literal $p$ in its body with $-\delta q \in \mathcal{T}_{T(D)+A}^+n$, or (3) there is a rule $s$ for $\neg q$ in $D$ where $+\sigma h(s) \in \mathcal{T}_{T(D)+A}^+n$, or (4) there is a rule for $\neg q$ in $A$ with body $B$ and $+\sigma B \subseteq \mathcal{T}_{T(D)+A}^+n$. (Some conditions are simpler than the inference rule for $-\delta$ might suggest because the superiority relation in $T(D)+A$ does not involve the rules for $q$ and $-\delta^*q$.) Consequently, $D+A \vdash -\Delta q$. In the first case, using the induction hypothesis, we have $D+A \vdash -\Delta q$ and $D+A \vdash +\Delta q$; hence, $D+A \vdash -\delta^*q$. In the second case, for each $r$, either $r$ is a defeater, or there is a literal $p$ in the body of $r$ such that $-\delta p \in \mathcal{T}_{T(D)+A}^+n$, or there is a rule $s$ for $\neg q$ in $D$ (corresponding to rule $n(r,s)$ in $T(D)$) with body $B$, where $+\sigma B \subseteq \mathcal{T}_{T(D)+A}^+n$ and $r \not\geq s$. By the induction hypothesis, either $r$ is a defeater, or there is a literal $p$ in the body of $r$ such that $D+A \vdash -\delta^*p$, or there is a rule $s$ for $\neg q$ in $D$ with body $B$, where $D+A \vdash +\sigma^*B$ and $r \not\geq s$. Similarly, using the induction hypothesis, every rule for $q$ in $A$ has a literal $p$ in its body with $D+A \vdash -\delta^*p$. Applying the inference rule for $-\delta^*$, we obtain $D+A \vdash -\delta^*q$.

In the third case, either $+\Delta h(s) \in \mathcal{T}_{T(D)+A}^+n$, or $+\sigma B \subseteq \mathcal{T}_{T(D)+A}^+n$, where $B$ is the body of $s$, and, for every rule $r$ for $q$ in $D$, either $-\delta p \in \mathcal{T}_{T(D)+A}^+n$ for some literal $p$ in the body of $r$ or $r \not\geq s$. If $+\Delta h(s) \in \mathcal{T}_{T(D)+A}^+n$ then $+\Delta B \subseteq \mathcal{T}_{T(D)+A}^+n$ and $s$ is strict. Using the induction hypothesis, $D+A \vdash +\Delta B$ and, hence, $D+A \vdash +\Delta q$ and, like case (1) above, $D+A \vdash -\delta^*q$. In the other case, by the induction hypothesis, $D+A \vdash +\sigma^*B$, and, for every rule $r$ for $q$ in $D$, either $D+A \vdash -\delta^*p$ for some literal $p$ in the body of $r$ or $r \not\geq s$. Applying the inference rule for $-\delta^*$ we conclude $D+A \vdash -\delta^*q$.

In the fourth case, using the induction hypothesis, there is a rule for $\neg q$ in $A$ with body $B$ and $D+A \vdash +\sigma^*B$. Applying the inference rule for $-\delta^*$ we conclude $D+A \vdash -\delta^*q$.

If $+\sigma^*q \in \mathcal{T}_{T(D)+A}^+(n+1)$ then either $+\Delta q \in \mathcal{T}_{T(D)+A}^+n$ (in which case $T(D)+A \vdash +\sigma q$) or $+\sigma^*B \subseteq \mathcal{T}_{T(D)+A}^+n$, where $B$ is the body of some strict or defeasible rule $r$ in $D+A$. In the latter case, for every rule $s$ for $\neg q$ in $D+A$ either $s \not\geq r$ or $-\delta^*p \in \mathcal{T}_{T(D)+A}^+n$ for some literal $p$ in the body of $s$. If $r \in A$ then $r$ is not inferior to any rule. So, by the induction hypothesis, $T(D)+A \vdash +\sigma B$, and, by the $+\sigma$ inference rule $T(D)+A \vdash +\sigma q$. If $r \in D$, by the induction hypothesis, $T(D)+A \vdash +\sigma B_r$, for every rule $n(r,s)$ for $\neg h(r)$ in $T(D)$, either $n(r,s) \not\geq p(r)$ or $T(D)+A \vdash -\delta^*p$ where $p$ occurs in the body of $n(r,s)$. Thus, using the inference rule for $+\sigma$, $T(D)+A \vdash +\sigma h(r)$. Applying the inference rule for $-\delta$ multiple times, for each strict or defeasible rule $s$ for $\neg q$ in $D$ we have $T(D)+A \vdash -\delta^*q$. Now, applying the inference rule for $+\sigma$, we have $T(D)+A \vdash +\sigma^*q$.

If $-\sigma^*q \in \mathcal{T}_{T(D)+A}^+(n+1)$ then $-\Delta q \in \mathcal{T}_{T(D)+A}^+n$ and for every strict or defeasible rule $r$ for $q$ in $D+A$, either $-\sigma^*p \in \mathcal{T}_{T(D)+A}^+n$ for some $p$ in the body of $r$ or there exists a rule $s$ for $\neg q$ in $D+A$ with body $B_s$, $+\delta^*B_s \subseteq \mathcal{T}_{T(D)+A}^+n$ and $s \not\geq r$. If $r \in A$ then $-\sigma^*p \in \mathcal{T}_{T(D)+A}^+n$ for some $p$ in the body of $r$ and hence, by the induction hypothesis, $T(D)+A \vdash -\sigma p$. If $r \in D$ then, for every strict or defeasible rule $p(r)$ in $T(D)$, either $-\sigma^*p \in \mathcal{T}_{T(D)+A}^+n$ for some $p$ in the body of $p(r)$ or there is a rule $n(r,s)$ with body $B_s$, where $+\delta^*B_s \subseteq \mathcal{T}_{T(D)+A}^+n$ and $n(r,s) \not\geq p(r)$, by the structure of $T(D)$. By the induction hypothesis, for every strict or defeasible rule $p(r)$ in $T(D)$, either $T(D)+A \vdash -\sigma p$ for some $p$ in the body of $p(r)$ or there is a rule $n(r,s)$ with body $B_s$, where $T(D)+A \vdash +\delta B$ and $n(r,s) \not\geq p(r)$. Application of the inference rule for $-\sigma$ gives us $T(D)+A \vdash -\sigma h(r)$ for each strict or defeasible rule $r$ for $q$ in $D$. Rules in $A$ for $q$ also fail, as mentioned above. Hence $T(D)+A \vdash -\sigma q$. 

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If $q \in \Sigma$ and $+\sigma q \in T_{(D)} \uparrow (n+1)$ then either $+\Delta q \in T_{(D)} \uparrow n$ (in which case $D+A \vdash +\sigma^* q)$, or $+\sigma B \subseteq T_{(D)} \uparrow n$ for some strict or defeasible rule for $q$ in $A$ with body $B$, or $+\sigma h(r) \in T_{(D)} \uparrow n$ for some strict or defeasible rule $r$ for $q$ in $D$. In the second case, by the induction hypothesis, $D+A \vdash +\sigma B$ and, applying the $+\sigma^*$ inference rule, $D+A \vdash +\sigma^* q$.

In the third case, if $+\sigma h(r) \in T_{(D)} \uparrow n$ then $p(r)$ is not a defeater, $+\sigma B_r \subseteq T_{(D)} \uparrow n$ where $B_r$ is the body of $r$ and for every rule $n(r,s)$ with body $B_s$ either for some literal $p$ in $B_n$, $-\delta p \in T_{(D)} \uparrow n$ or $n(r,s) \not\sim p(r)$. Using the construction of $T(D)$, $r$ is not a defeater, $+\sigma B_r \subseteq T_{(D)} \uparrow n$ where $B_r$ is the body of $r$ and for every rule $s$ for $\sim q$ with body $B_s$ in $D$ either for some literal $p$ in $B'$, $-\delta p \in T_{(D)} \uparrow n$ or $s \not\sim r$. Using the induction hypothesis, $D+A \vdash +\sigma^* B$, and for every rule for $\sim q$ in $D$ either for some literal $p$ in the body $D+A \vdash -\delta^* p$ or $s \not\sim r$. Note also that no rule $s$ for $\sim q$ in $A$ can be superior to $r$. Applying the inference rule for $+\sigma^*$ to this statement, we obtain $D+A \vdash +\sigma^* q$.

If $q \in \Sigma$ and $-\sigma q \in T_{(D)} \uparrow (n+1)$ then $-\Delta q \in T_{(D)} \uparrow n$ (and, consequently, $D+A \vdash -\Delta q$) and for every strict or defeasible rule $r$ for $q$ in $A$ with body $B$ there is $p$ in $B$ with $-\sigma p \in T_{(D)} \uparrow n$, and, for every rule $r$ for $q$ in $D$, $-\sigma h(r) \in T_{(D)} \uparrow n$. From $-\sigma h(r)$ either $r$ is a defeater, or there is a literal $p$ in the body of $r$ such that $-\sigma p \in T_{(D)} \uparrow n$, or there is a rule $s$ for $\sim q$ in $D$ (corresponding to rule $n(r,s)$ in $T(D)$) with body $B_s$ where $+\delta B_s \subseteq T_{(D)} \uparrow n$ and $s > r$. By the induction hypothesis, either $r$ is a defeater, or there is a literal $p$ in the body of $r$ such that $D+A \vdash -\sigma^* p$, or there is a rule $s$ for $\sim q$ in $D$ with body $B_s$ where $D+A \vdash +\delta^* B_s$ and $s > r$. Applying the inference rule for $-\sigma^*$, we obtain $D+A \vdash -\sigma^* q$.

\[\square\]

Combining Theorems 25 and 26 we have Theorem 12.

Simulation of Team Defeat wrt Addition of Rules

The same theory $D$ and addition $A$ as in Example 10 demonstrates that the simulation of $DL(\emptyset)$ by $DL(\emptyset^*)$ wrt addition of facts exhibited in (Maher 2012) does not extend to addition of rules.

Example 27

Let $D$ consist of the rules

$$
\begin{align*}
 r_1 & : \Rightarrow p \\
 r_2 & : \Rightarrow \neg p
\end{align*}
$$

and let $A$ be the rule

$$
\Rightarrow p
$$

Then $D + A \vdash \neg \delta p$.

The transformation presented in (Maher 2012) simulates $D$ wrt addition of facts with the following theory $D'$.
Theorem 29

Let \( \Sigma \) be the language of \( D+A \). Note, that, employing Lemma 18, \( T(D)+A \vdash +\partial^{*}true(q) \) iff \( T(D)+A \vdash -\Delta q \) iff \( D+A \vdash -\Delta q \). Because \( T(D)+A \vdash -\partial^{*}q \), we can essentially ignore the rules \( supp(q) \), which are only included for the simulation of \( \text{DL}(\delta) \) by \( \text{DL}(\partial^{*}) \).

Suppose \( +\partial q \in T_{D+A} \uparrow (n+1) \). Then either \( +\Delta q \in T_{D+A} \uparrow n \) (in which case \( T(D)+A \vdash +\partial^{*}q \)), or \( -\Delta \neg q \in T_{D+A} \uparrow n \) and there is a non-empty team of strict or defeasible rules for \( q \).
such that $+\partial B_r \subseteq \mathcal{T}_{D+A} \uparrow n$ for each body $B_r$ of each rule $r$ and every rule $s$ for $\sim q$ either has a body that fails in $\mathcal{T}_{D+A} \uparrow n$ or $s < t$ for some rule $t$ in the team. \( t \notin A \) because rules in $A$ do not participate in the superiority relation. Then, by the induction hypothesis, $T(D) + A \vdash -\Delta \sim q$, $T(D) + A \vdash +\partial^* B_r$ for each rule $r$ in the team, and for every rule $s$ for $\sim q$ either its body fails in $T(D) + A$ or there is a rule $t$ in the team and $t > s$. If $s \in A$ then its body $B_s$ fails in $T(D) + A$. If $s \in D$ then either $T(D) + A \vdash +\partial^* \text{fail}(s)$ or $T(D) + A \vdash +\partial^* d(s,t)$; in either case, $T(D) + A \vdash +\partial^* d(s)$. Considering $T(D)$ and the inference rule for $+\partial^*$, we have $T(D) + A \vdash +\partial^* \text{true} (\sim q)$. By Lemma 28, $T(D) + A \vdash +\partial^* \text{true} (\sim q)$. Hence, the body of $s(q)$ is proved. Because $\triangleright$ is acyclic, there is a rule in the team for $q$ that is not inferior to any rule in the team for $\sim q$. Hence this rule $r'$ is not defeated, so $d (r')$ fails, and hence the rule $s(\sim q)$ in $T(D)$ (from point 6) fails. Hence all rules for $\sim q$ fail, with the possible exception of $o(\sim q)$. However $s(q) > o(\sim q)$ and hence, applying the $+\partial^*$ inference rule, $T(D) + A \vdash +\partial^* q$.

Suppose $-\partial q \in \mathcal{T}_{D+A} \uparrow (n + 1)$. Then $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (and hence $T(D) + A \vdash -\Delta q$) and either (1) $+\Delta \sim q \in \mathcal{T}_{D+A} \uparrow n$ (in which case $T(D) + A \vdash +\partial^* d(s)$) or (2) every rule $r$ for $q$ fails, or (3) there is a rule $s$ for $\sim q$ with body $B_s$ such that $+\partial B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and, for every strict or defeasible rule $t$ for $q$, either $t$ fails in $\mathcal{T}_{D+A} \uparrow n$, or $t \neq s$. In case (2), the rules in $A$ for $q$ fail and, by the induction hypothesis and the inference rule for $-\partial^*$, the rules $r$ in $A$ for $q$ fail and, $T(D) + A \vdash -\partial^* \text{one}(q)$ and hence $T(D) + A \vdash -\partial^* q$. In case (3), by the induction hypothesis, there is a rule $s$ for $\sim q$ with body $B_s$ such that $T(D) + A \vdash +\partial^* B_s$ and for every strict or defeasible rule $t$ for $q$, either $t$ fails in $T(D) + A$, or $t \neq s$, or, for every $t$, $T(D) + A \vdash +\partial^* d (s, t)$ (since, via Lemma 28, we also have $T(D) + A \vdash +\partial^* d(s)$) and hence $T(D) + A \vdash -\partial^* q$.

Suppose $q \in \Sigma$ and $+\partial^* q \in \mathcal{T}_{T(D)+A} \uparrow (n + 1)$. Then either (1) $+\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash +\partial q$), or (2) every rule $r$ for $q$ fails, or (3) $+\partial^* \text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, and $+\partial^* d(r)$ occurs in $\mathcal{T}_{T(D)+A} \uparrow n$, for each rule $r$ for $q$ in $D$. In both cases (2) and (3) we must have, for any rule $s$ for $\sim q$ in $A$, for some $p$ in the body $B_s$ of $s$, $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, $D + A \vdash -\partial p$ for each such $p$.

In case (2), by the induction hypothesis, $D + A \vdash B_r$. Also, in case (2), the rule $o(\sim q)$ must fail. Consequently, every rule $s$ for $\sim q$ in $D$ fails in $\mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, every rule $s$ for $\sim q$ in $D$ fails in $D + A$. Now, applying the inference rule for $+\partial$, $D + A \vdash +\partial q$.

In case (3) there must be a strict or defeasible rule $r$ for $q$ in $D$ with body $B_r$ such that $+\partial^* B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, using the rules for $d (s)$ and $d (s,t)$, for every rule $s$ for $\sim q$, either the body $B_s$ of $s$ fails or there is a strict or defeasible rule $t$ for $q$ with body $B_t$ such that $+\partial^* B_t \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $t < s$. By the induction hypothesis, $D + A \vdash +\partial B_s$, and, for every rule $s$ for $\sim q$ in $D$, either $D + A \vdash -\partial B_s$ or there is a strict or defeasible rule $t$ for $q$ with body $B_t$ such that $D + A \vdash +\partial B_t$ and $t > s$. As noted above, for any rule $s$ for $\sim q$ in $A$, $B_s$ fails in $D + A$. Hence, by the inference rule for $+\partial$, $D + A \vdash +\partial q$.

If $q \in \Sigma$ and $-\partial^* q \in \mathcal{T}_{T(D)+A} \uparrow (n + 1)$ then, using the inference rule for $-\partial^*$ and the structure of $T(D)$, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ and either (a) $+\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash -\partial^* q$), or (b) $-\partial^* \text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, or (c) $+\partial^* d(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some rule $s$ for $\sim q$ in $D$, or (d) $+\partial^* \text{true} (\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$ and $+\partial^* d(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for every rule $r$ for $q$ in $D$, or (e) for some $s$ for $\sim q$ in $A$, $+\partial^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$.

If (b) $-\partial^* \text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$ then for every strict or defeasible rule for $q$ in $D$ fails. Applying the induction hypothesis and the inference rule for $-\partial$, we have $D + A \vdash -\partial q$. If (c)
\[ -\partial^*d(s) \in \mathcal{T}_{T(D)+A} \uparrow n \] for some rule \( s \) for \( \sim \) in \( D \), then there is no strict or defeasible rule \( r_s \) for \( q \) that defeats \( s \). If (d) then there is a rule \( s \) for \( \sim \) with body \( B_s \) such that \( +\partial^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n \) and every rule \( r \) for \( q \) in \( D \) is defeated by a strict or defeasible rule for \( \sim \). In both cases (c) and (d), applying the induction hypothesis and the inference rule for \( -\partial \), we have \( D + A \vdash -\partial q \). If (e) then, by the induction hypothesis, \( D + A \vdash +\partial B_s \) and hence, applying the inference rule for \( -\partial \), \( D + A \vdash -\partial q \).

\[ \square \]

**Theorem 30**

The logic \( \mathbf{DL}(\delta) \) can be simulated by \( \mathbf{DL}(\delta^*) \) with respect to addition of rules.

**Proof**

Let \( \Sigma \) be the language of \( D+A \). Note that, for any \( q \in \Sigma(D) \), \( T(D)+A \vdash +\sigma^*\sim true(q) \) and, employing Lemma 18, \( T(D)+A \vdash +\delta^*\sim true(q) \) if \( T(D)+A \vdash -\Delta q \) if \( D+A \vdash -\Delta q \). Also note that \( T(D)+A \vdash -\delta^*g \), \( T(D)+A \vdash +\sigma^*g \), and \( T(D)+A \vdash +\delta^*g \), where \( g \) is the proposition used in part 7 of Definition 13.

Suppose \( +\delta q \in \mathcal{T}_{D+A} \uparrow (n+1) \). Then either \( +\delta q \in \mathcal{T}_{D+A} \uparrow n \), or \( -\delta \sim q \in \mathcal{T}_{D+A} \uparrow n \) and there is a non-empty team of strict or defeasible rules for \( q \) such that \( +\delta B_r \subseteq \mathcal{T}_{D+A} \uparrow n \) for each body \( B_r \) of each rule \( r \) and every rule \( s \) for \( \sim \) either has a body that fails in \( \mathcal{T}_{D+A} \uparrow n \) or \( s < t \) for some rule \( t \) in the team. Then, by the induction hypothesis, either \( T(D) + A \vdash +\delta \) or \( \mathcal{T}_{D+A} \uparrow n \) (in which case \( T(D) + A \vdash +\delta^*q \)), or \( T(D) + A \vdash -\delta \sim q \), \( T(D) + A \vdash +\delta^*B_r \) for each rule \( r \) in the team, and for every rule \( s \) for \( \sim \) with body \( B_s \), either \( T(D) + A \vdash -\sigma B_s \) or there is a rule \( t \) in the team and \( t > s \). If \( s \in A \) then \( T(D) + A \vdash -\sigma B_s \). If \( s \in D \) then either \( T(D) + A \vdash +\delta^*fail(s) \) or \( T(D) + A \vdash +\delta^*d(s,t) \); in either case, \( T(D) + A \vdash +\delta^*d(s) \).

Considering \( T(D) \), and the inference rule for \( +\delta^* \) we have \( T(D) + A \vdash +\delta^*one(q) \). By Lemma 18, \( T(D) + A \vdash +\delta^*\sim true(\sim q) \). Hence, the body of \( s(q) \) is proved. Because \( > \) is acyclic, there is a rule in the team for \( q \) that is not inferior to any rule in the team for \( \sim q \). Hence this rule \( r' \) is not defeated, so \( d(r') \) fails, and hence the rule for \( q \) in \( T(D) \) from point 6 fails. Similarly, \( d_p(r', s) \) fails, and hence the rules for \( q \) in \( T(D) \) from point 7 fail. Hence all rules for \( q \) fail, with the possible exception of \( o(\sim q) \). However \( s(q) > o(\sim q) \) and hence, applying the \( +\delta^* \) inference rule, \( T(D) + A \vdash +\delta^*q \).

Suppose \( -\delta q \in \mathcal{T}_{D+A} \uparrow (n+1) \). Then \( -\delta q \in \mathcal{T}_{D+A} \uparrow n \) and (hence \( T(D) + A \vdash -\delta q \)) and either (1) \( +\delta \sim q \in \mathcal{T}_{D+A} \uparrow n \) (in which case \( T(D) + A \vdash -\delta^*g \)), or (2) every rule \( r \) for \( q \) contains a body literal \( p \) with \( -\partial p \in \mathcal{T}_{D+A} \uparrow n \), or (3) there is a rule \( s \) for \( q \) with body \( B_s \) such that \( +\sigma B_s \subseteq \mathcal{T}_{D+A} \uparrow n \) and, for every strict or defeasible rule \( t \) for \( q \), either \( t \) fails in \( \mathcal{T}_{D+A} \uparrow n \), or \( t \neq s \). In case (2), the rules \( r \) in \( A \) for \( q \) fail and, by the induction hypothesis and the inference rule for \( -\delta^* \), the rules \( r \) in \( A \) for \( q \) fail in \( T(D) + A \), so \( T(D) + A \vdash -\delta^*one(q) \) and hence \( T(D) + A \vdash -\delta^* \). In case (3), by the induction hypothesis, there is a rule \( s \) for \( \sim q \) with body \( B_s \) such that \( T(D) + A \vdash +\sigma^*B_s \) and for every strict or defeasible rule \( t \) for \( q \), either \( t \) fails in \( T(D) + A \), or \( t \neq s \). If \( s \in A \) then \( t \neq s \), for every \( t \), and hence \( T(D) + A \vdash -\delta^*q \). If \( s \in D \) then \( T(D) + A \vdash -\delta^*d(s,t) \) (since, via Lemma 18, we also have \( T(D) + A \vdash -\delta^*true(q) \)). Using the \( -\delta^* \) inference rule, \( T(D) + A \vdash -\delta^*d(s) \). The bodies of rules from point 7 of the transformation also fail (wrt \( \delta^* \)), because of the presence of \( g \). Hence \( T(D) + A \vdash -\delta^*q \).

Suppose \( q \in \Sigma \) and \( +\delta^*q \in \mathcal{T}_{T(D)+A} \uparrow (n+1) \). Then either (1) \( +\delta q \in \mathcal{T}_{T(D)+A} \uparrow n \) (in which case \( D + A \vdash +\delta q \)), or else \( -\delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n \) and either (2) there is a strict or
defeasible rule $r$ for $q$ in $A$ where $+\delta^*B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and for all rules for $\neg q$ in $T(D)+A$, the body of the rule contains a literal $p$ with $-\sigma^*p \in \mathcal{T}_{T(D)+A} \uparrow n$, or (3) each of $+\delta^*\text{one}(q)$, $+\delta^*\text{false}(-q)$, and $+\delta^*\text{d}(s)$ occurs in $\mathcal{T}_{T(D)+A} \uparrow n$, for each rule $s$ for $\neg q$ in $D$.

Hence, in case (3), there is a strict or defeasible rule $r$ for $q$ with body $B_r$ such that $+\delta^*B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, for every rule $s$ for $\neg q$, either $-\sigma^*p \in \mathcal{T}_{T(D)+A} \uparrow n$, for some $p$ in the body $B_s$ of $s$, or there exists $t$ in $D$ for $q$ with $+\delta^*B_t \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $t > s$. By the induction hypothesis, $D + A \vdash -\Delta q$, $D + A \vdash +\delta B_r$, and, for every rule $s$ for $\neg q$, either $D + A \vdash -\sigma B_s$ or $D + A \vdash +\delta B_t$ and $t > s$. By the inference rule for $+\delta$, $D + A \vdash +\delta q$.

In case (2), using the structure of $T(D)$, for the rules $\text{supp}(-q)$, originating from some rule $s$ for $\neg q$ in $D$, either for some $p$ in $B_s$, $-\sigma^*p \in \mathcal{T}_{T(D)+A} \uparrow n$ or, for some $t$, $-\sigma^*d_\sigma(t, s) \in \mathcal{T}_{T(D)+A} \uparrow n$ (and, hence, $+\delta^*B_t \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $t > s$). Now, by the induction hypothesis, $D + A \vdash +\delta B_t$; $D + A \vdash -\Delta q$; for all rules for $\neg q$ in $A$, the body of the rule contains a literal $p$ with $D + A \vdash -\sigma^*p$; and for all rules for $\neg q$ in $D$, either the body of the rule contains a literal $p$ with $D + A \vdash -\sigma^*p$ or there is a rule $t$ for $q$ in $D$ with $t > s$ and $D + A \vdash +\delta B_t$. Applying the inference rule for $+\delta$, $D + A \vdash +\delta q$.

If $q \in \Sigma$ and $-\delta^*q \in \mathcal{T}_{T(D)+A} \uparrow (n + 1)$ then, using the inference rule for $-\delta^*$ and the structure of $T(D)+A$, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (and, hence, $D + A \vdash -\Delta q$) and either $+\Delta \neg q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash -\delta q$), or else for every rule $r$ for $q$ in $A$, there is a literal $p$ in $B_r$ such that $-\delta^*p \in \mathcal{T}_{T(D)+A} \uparrow n$ and either (1) $-\delta^*\text{true}(-q) \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash +\Delta \neg q$ and hence $D + A \vdash -\delta q$), or (2) $-\delta^*\text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, or (3) $-\delta^*d(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some rule $s$ for $\neg q$ in $D$. Or (4) $+\sigma^*\text{one}(-q) \in \mathcal{T}_{T(D)+A} \uparrow n$ and $+\sigma^*d(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for every rule $r$ for $q$ in $D$, or (5) there is a rule $s$ for $\neg q$ in $A$ and $+\sigma^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$. Or (6) the body of a rule $\text{supp}(-q)$ is supported for some rule $s$ for $\neg q$ (that is, $+\sigma^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, for each rule $r$ for $q$, $+\sigma^*d_\sigma(r, s) \in \mathcal{T}_{T(D)+A} \uparrow n$).

For (2) and (3), by the induction hypothesis, for every rule $r$ for $q$ in $A$, there is a literal $p$ in $B_r$ such that $D + A \vdash -\delta p$. If (2) $-\delta^*\text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$ then every strict or defeasible rule for $q$ in $D$ fails. Applying the induction hypothesis and the inference rule for $-\delta$, we have $D + A \vdash -\delta q$. If (3) $-\delta^*d(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some rule $s$ for $\neg q$ in $D$, then $+\sigma^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, where $B_s$ is the body of $s$, and for every strict or defeasible rule $r$ for $q$ with body $B$ either $-\delta^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ or $+\sigma^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, where $B_s$ is the body of $s$, and $r \neq s$. Applying the induction hypothesis, $D + A \vdash +\sigma B_s$ and, for every rule $r$ for $q$, $D + A \vdash -\delta B_r$ or $D + A \vdash +\sigma B_s$ and $r \neq s$. Hence, by the inference rule for $-\delta$, $D + A \vdash -\delta q$.

If (4) then there is a rule $s$ for $\neg q$ with body $B_s$ such that $+\sigma^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and for every rule $r$ for $q$ in $D$ either there is a literal $p$ in the body of $r$ such that $-\delta^*p \in \mathcal{T}_{T(D)+A} \uparrow n$ or there is a rule $s'$ for $\neg q$ with body $B'$ such that $+\sigma^*B' \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $s' > r$. Applying the induction hypothesis, for every rule $r$ for $q$ in $D$ either there is a literal $p$ in the body of $r$ such that $D + A \vdash -\delta p$ or there is a rule $s'$ for $\neg q$ with body $B'$ such that $D + A \vdash +\sigma B'$ and $s' > r$, 1 follows, by the inference rule for $-\delta$, that $D + A \vdash -\delta q$.

If (5) then, by the induction hypothesis, $D + A \vdash +\sigma B_s$ and, since $s$ is not inferior to any rule, the inference rule for $-\delta$ gives us $D + A \vdash -\delta q$.

In case (6), since $+\sigma^*d_\sigma(r, s) \in \mathcal{T}_{T(D)+A} \uparrow n$, we must have $+\sigma^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and either there is a literal $p$ in $B_s$ such that $-\delta^*p \in \mathcal{T}_{T(D)+A} \uparrow n$ or $r \neq s$. By the induction hypothesis, $D + A \vdash +\sigma B_s$ and, for every rule $r$ for $q$ in $D$ either there is a literal $p$ in $B_r$ such that $D + A \vdash -\delta p$ or $r \neq s$. By the $-\delta$ inference rule, $D + A \vdash -\delta q$.

Suppose $+\sigma^* \in \mathcal{T}_{D+A} \uparrow (n + 1)$. Then either $+\Delta q \in \mathcal{T}_{D+A} \uparrow n$, or there is a strict or defeasible rule $r$ for $q$ such that $+\sigma B_r \subseteq \mathcal{T}_{D+A} \uparrow n$ where $B_r$ is the body of $r$ and every rule $s$
for $\sim q$ has a body with a literal $p$ such that $-\delta p \in \mathcal{T}_{D+A} \uparrow n$ or $s \not> r$. Then, by the induction hypothesis, either $T(D) + A \vdash +\Delta q$ (in which case $T(D) + A \vdash +\sigma^* q$), or $T(D) + A \vdash +\sigma^* B_r$, and every rule $s$ for $\sim q$ has a body with a literal $p$ such that $-\delta^* p \in \mathcal{T}_{D+A} \uparrow n$ or $s \not> r$. If $r \in A$ then $r$ is not inferior to any rule and, by the inference rule for $+\sigma^*$, $T(D) + A \vdash +\sigma^* q$. If $r \in D$ then, by the $+\sigma^*$ inference rule, $T(D) + A \vdash +\sigma^* B_r$. Furthermore, again by the $+\sigma^*$ inference rule, for every $s$ for $\sim q$ in $D$, $T(D) + A \vdash +\sigma^* d_s(s, r)$, since $a(s, r) \not> b(s, r)$ if $s \not> r$. Note that there is no superiority relation between the rules in $T(D)$ for $q$ and $\sim q$. Hence, applying the inference rule for $+\sigma^*$, $T(D) + A \vdash +\sigma^* q$.

Suppose $-\sigma q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ and either all rules for $\sim q$ with body $B_r$ such that $-\delta B_r \subseteq \mathcal{T}_{D+A} \uparrow n$ and $s \not> r$. (Note that, for $r \in A$, only the first possibility can apply.) Then, by the induction hypothesis, $T(D) + A \vdash -\Delta q$ and either all rule $s$ for $\sim q$ contains a body literal $p$ such that $T(D) + A \vdash -\sigma^* p$, or there is a rule $s$ for $\sim q$ with body $B_r$ such that $T(D) + A \vdash +\sigma^* B_r$ and $s \not> r$. (In particular, every rule for $q$ in $A$ contains a body literal $p$ with $T(D) + A \vdash -\sigma^* p$.) If all rules for $q$ in $D$ fall in the former case, we have $-\sigma^* w(q)$, and all rules $supp(q)$ fail. Otherwise, there is an $s$ that is not inferior to any rule for $q$ and hence $T(D) + A \vdash -\sigma^* d(s, r)$ and $T(D) + A \vdash -\sigma^* d(s)$. Similarly, $T(D) + A \vdash -\sigma^* d_s(s, r)$. In either case, all rules for $q$ fail, and hence $T(D) + A \vdash -\sigma^* q$.

Suppose $q \in \Sigma$ and $+\sigma q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then either (1) $+\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (in which case $D + A \vdash +\sigma q$), or else either (2) for some such rule $r$ for $q$ in $A$, $+\sigma^* B_r \subseteq \mathcal{T}_{D+A} \uparrow n$, or (3) $+\sigma^* w(q) \in \mathcal{T}_{D+A} \uparrow n$, and $+\sigma^* d(s) \in \mathcal{T}_{D+A} \uparrow n$, and, for each rule $s$ for $\sim q$ in $D$, or (4) for some strict or defeasible rule $r$ for $q$ in $D$, $+\sigma^* B_r \subseteq \mathcal{T}_{D+A} \uparrow n$ and $+\sigma^* d_s(s, r) \in \mathcal{T}_{D+A} \uparrow n$, for each rule $s$ for $\sim q$ in $D$.

In case (2), by the induction hypothesis, $D + A \vdash +\sigma B_r$ and hence, by the inference rule for $\sigma$, $D + A \vdash +\sigma q$.

In case (3), there is a strict or defeasible rule $r$ for $q$ with body $B_r$ such that $+\sigma^* B_r \subseteq \mathcal{T}_{D+A} \uparrow n$, and, for every rule $s$ for $\sim q$ in $D$, either $-\delta B_s \subseteq \mathcal{T}_{D+A} \uparrow n$, for some $p$ in the body $B_r$ of $s$ or there is a rule $t$ for $q$ with body $B_t$ such that $+\sigma^* B_t \subseteq \mathcal{T}_{D+A} \uparrow n$, and $s \not> t$. By the induction hypothesis, $D + A \vdash +\sigma B_r$, and, for every rule $s$ for $\sim q$, either $D + A \vdash -\delta B_s$ or there is a rule $t$ for $q$ with body $B_t$ such that $D + A \vdash +\sigma B_t$ and $s \not> t$. Because $\mathcal{I}$ is acyclic, there is a rule $t$ for $q$ such that $D + A \vdash +\sigma B_t$ and, for every rule $s$ for $\sim q$ either $D + A \vdash -\delta B_s$ or $s \not> t$. By the inference rule for $+\sigma$, $D + A \vdash +\sigma q$.

In case (4), there is a strict or defeasible rule $r$ for $q$ with body $B_r$ such that $+\sigma^* B_r \subseteq \mathcal{T}_{D+A} \uparrow n$ and, for each rule $s$ for $\sim q$ in $D$, either $-\delta p \in \mathcal{T}_{D+A} \uparrow n$, for some $p$ in the body $B_r$ of $s$, or $s \not> r$. By the induction hypothesis, $D + A \vdash +\sigma B_r$, and, for each rule $s$ for $\sim q$, either $D + A \vdash -\delta p$ or $s \not> r$. By the $+\sigma^*$ inference rule, $D + A \vdash +\sigma q$.

If $q \in \Sigma$ and $-\sigma q \in \mathcal{T}_{D+A} \uparrow (n+1)$, then, using the inference rule for $-\sigma^*$ and the structure of $T(D)$, $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (and hence $D + A \vdash -\Delta q$), and either (1) $-\Delta w(q)$, or (2) for each strict or defeasible rule $r$ for $q$ in $A$, there is a literal $p$ in $B_r$ such that $-\sigma^* p \in \mathcal{T}_{D+A} \uparrow n$ and for each strict or defeasible rule $r$ for $q$ in $D$, either there is a literal $p$ in $B_r$ such that $-\sigma^* p \in \mathcal{T}_{D+A} \uparrow n$, or there is a rule $s$ for $\sim q$ in $D$ such that $+\delta^* B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and $s \not> r$. By the induction hypothesis, in case (2), $D + A \vdash -\sigma B_r$ for the rules $r$ in $A$ and, for rules $r$ in $D$, either $D + A \vdash -\sigma B_r$ or there is a rule $s$ for $\sim q$ in $D$ such that $D + A \vdash +\delta^* B_s$ and $s \not> r$. Applying the $-\sigma$ inference rule, $D + A \vdash -\sigma q$. □
Combining Theorems 29 and 30, we have Theorem 14.

References


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